Duality theories for the p-primary étale theory

$\boxed{20/10/2022}$ Preprint Seminar

Series of papers of the same title:
— I, Kato, 1985, smooth equichar p
— II, Kato, 1986, l.f.t. equi-char p
— III, Kato-Suzuki, 2019, mixed char (w.p.) (smooth)

Notation: K henselian d.v. field, O_K, k, (char $= 0$) (char $= p$).

0. Motivation

l-primary case: $\Lambda = \mathbb{Z}/l^n$, $\Lambda(s) = \Lambda \otimes \mu_{l^n}^{\otimes s}$, $l \neq p = \mathrm{char}(k)$.

- equichar case (Poincaré duality)

Y (connected) smooth quasi-proj dim d, $\pi: Y \to k = \bar{k}$ $\Rightarrow$ $\exists$ perfect pairing

$$H^i_{ét,c}(Y, \Lambda(s)) \times H^{2d-i}_{ét}(Y, \Lambda(d-s)) \to H^{2d}_{ét,c}(Y, \Lambda(d)) \xrightarrow[\cong]{Tr} (\mathbb{Z}/l^n)$$

- relative version (Verdier duality)

$X \xrightarrow{\pi} S$ sm. quasi-proj. fiber of dim d $\Rightarrow$ $\exists$ perfect duality, i.e.

$\exists$ derived duality $R\pi_* \Lambda(d-s) \cong R\mathcal{H}om(R\pi_! \Lambda(s), \mathbb{Z}/l^n[-2d])$

(In general form: $R\pi_* D(F) \cong D(R\pi_! F)$, F constructible $\mathbb{Z}/l^n$-sh)

- mixed char. (on vanishing cycles $R\Psi\Lambda$, Illusie)

$X \to O_K$ sm (or only l.f.t. separated) dim d

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & U \arrow[l, hook', "j"'] \arrow[d, dash] \\ k & O_K & K \end{tikzcd}$$

$R\Psi = \bar{i}^* R\bar{j}_* : (U_{\bar\eta})_{ét} \to (Y_{\bar s})_{ét}$ $\Rightarrow$ $\exists$ (perfect) duality: $R\Psi\Lambda$ and $R\Psi\Lambda^\vee$.

p-primary case: What about $l = p$? $\Lambda = \mathbb{Z}/p^n$, $\Lambda(s) = \Lambda \otimes \mu_{p^n}^{\otimes s}$, $Y \xrightarrow{\text{smooth}} k$.

- Problem: μ_p is trivial on $Y_{ét}$, but non-trivial on Y_{fppf}.

$R\varepsilon_* \mu_p \cong \nu(1)[-1]$, where $\varepsilon: Y_{fppf} \to Y_{ét}$; for $\nu(1)$, see below.

- Milne: $\exists$? duality between $H^i_{fppf}(Y, \mu_p) \times H^{5-i}_{fppf}(Y, \mu_p) \to \mathbb{Z}/p$ (Y proper surface)

Artin's observation: say Y proper surface, if k is finite, no problem (Milne),

if $k = \bar{k}$, then the infiniteness cause a problem. However,

$$H^i_{fppf}(Y, \mu_p) = \begin{cases} \text{finite} & i=0 \\ \text{finite} & 1 \\ \text{finite} + \text{v.s.} & 2 \\ \text{finite} + \text{v.s.} & 3 \\ \text{finite} & 4 \end{cases}$$

So expect a duality ~~for~~ with $i+j=4$ for the finite part,

with $i+j=5$ for the vector space part.

• Milne (Duality for flat coh. of surface, 1976): Y proper smooth dim d.

$$R\pi_* \nu(r) \xrightarrow{\sim} R\mathcal{H}om(R\pi_* \nu(d-r), \mathbb{Z}/p[-d]) \quad \text{in } D((\mathrm{Perf}/S)_{et}, \mathbb{Z}/p)$$

$R\varepsilon_* \mu_p = \nu(1)[-1]$

$\Rightarrow$ ~~$R\pi_*$~~ $R\pi_* \mu_p \cong (R\pi_* \mu_p)^\vee[-4]$ if for flat cohomology

$d=2$

where the $(-)^\vee = R\mathcal{H}om_{\mathbb{F}_p}(-, \mathbb{Z}/p)$

Note that the duality happens on some big perfect étale site.

Strategy: $\nu(d) = [\Omega^d \xrightarrow{C^{-1}} \Omega^d]$ for étale topology, use a result of [Breen]

1. [Kato, I] influenced by Milne's work, worked with a relative version of the perfect site — the relatively perfect site.

(Equi-char. setting)

Def. $\begin{array}{c} Y' \\ \downarrow \\ Y/\mathbb{F}_p \end{array}$ is relatively perfect if

$$\begin{tikzcd} Y' \arrow[r, "\mathrm{Frob}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\ Y \arrow[r, "\mathrm{Frob}"'] & Y \end{tikzcd}$$

is cartesian.

Ex. étale $\Rightarrow$ rel. perfect $\Rightarrow$ formally étale $\overset{\text{l.f.t.}}{\Longrightarrow}$ étale.

Def. Y_{RP} the site of schemes rel. perf $/Y$ + étale topology, ringed: $\mathcal{O}: T \mapsto \mathcal{O}(T)$.

Hyp. Suppose $Y/\mathbb{F}_p$ (reduced) and $\exists$ affine covering $\{U_\lambda\}_\lambda$ with p-base of r elements $(b_1, \dots, b_r)$, i.e. $\mathcal{O}(U_\lambda) \xrightarrow{(-)^p} \mathcal{O}(U_\lambda)$ is locally free of rk p^r with basis $b_1^{i_1} \cdots b_r^{i_r} =: \underline{b}^I$ for $I = (i_1, \dots, i_r) \in [0, p-1]^r$ $:= \{0, \dots, p-1\}^r$

Def/Prop. ① $\Omega^q_{Y/\mathbb{F}_p}$ is loc. free $\mathcal{O}_Y$-module of rank r. locally with basis $(db_1, \dots, db_r)$.

② Define $\nu(q) \subseteq \Omega^q_{Y/\mathbb{F}_p}$ the sub-étale-sheaf generated by $\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}$, $y_i \in \mathcal{O}_Y^*$.

③ Have exact sequences of étale sheaves

$$\begin{tikzcd}
0 \arrow[r] & \nu(q) \arrow[r] & Z^q_Y \arrow[r, "C-1"] \arrow[d, "C"] & \Omega^q_Y \arrow[r] \arrow[d, "\text{quotient}"] & 0 \\
0 \arrow[r] & \nu(q) \arrow[r] & \Omega^q_Y \arrow[r, "C^{-1}-1"] & \Omega^q_Y/B^q_Y \arrow[r] & 0
\end{tikzcd}$$

④ Also, by easy computation, Z^q_Y & B^q_Y are locally free $\mathcal{O}_Y$-sheaves of finite rank, under the $\mathcal{O}_Y$-mod structure $\mathcal{O}_Y \xrightarrow{(-)^p} \mathcal{O}_Y \underset{\text{usual action}}{\circlearrowright} Z^q_Y, B^q_Y \subseteq \Omega^q_Y$.

⑤ Denote $\mathbb{G}_a : T \longmapsto \mathcal{O}(T) \in Ab$.

Thm (Kato) As $\mathbb{F}_p$-sheaves on Y_{RP},

$$R\mathcal{H}om_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \mathcal{H}om_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a)$$

$$R\mathcal{H}om_{\mathbb{F}_p}(M^{RP}, \nu(r)) = (M^\vee)^{RP}[-1]$$

$$R\mathcal{H}om_{\mathbb{F}_p}(\nu(q), \nu(r)) = \nu(r-q)[0]$$

Here, M locally free $\mathcal{O}_{Y_{Zar}}$-mod. of finite rank, $M^\vee$ coherent dual of $M = \mathcal{H}om_{\mathcal{O}_{Y_{Zar}}}(M, \Omega^r_Y)$, by $(-)^{RP}$ meaning natural extension to the site Y_{RP}.

Rmk Different shifts for $\nu(q)$-type sheaves and Vect. bundle-sheaves.

Rmk. 1) Crucial step: on $Y_{Ét}$. $\mathcal{H} = \mathcal{H}om_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a)$

$$\mathcal{E}xt^{2j+1}_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = 0, \quad j \geq 0$$

$$\mathcal{E}xt^{2j}_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \mathcal{H} / \langle \mathrm{Frob}^{v_p(j)+1} \rangle, \quad j > 0.$$

← bilateral ideal

[Breen] IHES

and on affines, by Breen-Deligne resolution, one gets

$$Ext^i_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a) = \varinjlim_n Ext^i_{\mathbb{F}_p, Y_{Ét}}(G^n(\mathbb{G}_a), \mathbb{G}_a)$$

see below for the Weil restriction functor G on $(Sch/_Y)$.

and $G^m(\mathbb{G}_a) \longrightarrow \mathbb{G}_a$ factors through some

$$\begin{array}{ccc} G^m(\mathbb{G}_a) & & \mathbb{G}_a \\ \cup \text{ direct summand} & \nearrow & \\ \mathbb{G}_a^{\oplus s} & \text{direct sum of } \mathrm{Frob}^m & \end{array}$$

so for $m \geq v_p(j)+1$, this kills $Ext^i_{\mathbb{F}_p, Y_{Ét}}(G^{\bullet}(\mathbb{G}_a), \mathbb{G}_a)$ in $Ext^i_{\mathbb{F}_p, Y_{Ét}}(G^{\bullet+m}(\mathbb{G}_a), \mathbb{G}_a)$.

Hence the vanishing of this $\varinjlim_n$.

2) One ingredient hidden in [Breen] is the verification of so-called

(A1') $\forall n$, $R[x_1, \dots, x_n] \xrightarrow[\sim]{R\text{-alg}} \mathrm{Hom}_T(\mathbb{G}_a^n, \mathcal{O})$

$x_i \longmapsto$ i^{th} projection to $\mathrm{Hom}_T(\mathbb{G}_a, \mathcal{O})$

(A2) $\forall n$, $H^q(\mathbb{G}_a^n, \mathbb{G}_a) = 0$ ← cohomology in the topos. $\mathbb{G}_a$ not necessarily representable.

on a ringed topos.

For verification on $Y_{Ét}$ for Y affine:

(A1') needs representability of $\mathbb{G}_a$ (by affine $\mathbb{A}^1_Y$)

(A2) needs it too, and $H^q_{fppf}(\overset{\mathbb{G}_a^n}{X}, \mathbb{G}_a) = H^q_{ét}(X, \mathbb{G}_a) = H^i_{Zar}(X, \mathbb{G}_a)$

by Hilbert 90, and $H^i_{Zar}(X, \mathbb{G}_a)$ by Serre's vanishing

since X is representable by the affine scheme $\mathbb{A}^n_Y$.

So in fact, our theorem holds if replacing Y_{RP} by some smaller site s.t. this condition holds.

2. Mixed char. case : formulation.

$\Lambda = \mathbb{Z}/p$. $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$, $\downarrow$ sm, $\dim = d$, $\mathcal{O}_K$

$[k : k^p] = p^{r_0}$

Put $r := r_0 + d$

2.1 [Bloch–Kato, p-adic étale coh.]

$\Lambda \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to \Lambda$ on U_{et}

$\rightsquigarrow$ $i^* j_* \mathbb{G}_m \to R^1\psi\Lambda(1)$ on Y_{et}

$\rightsquigarrow$ $(i^* j_* \mathbb{G}_m)^{\otimes q} \to R^q\psi\Lambda(q)$ symbol map (induced by cup product)

$\{x_1, \ldots, x_q\} \longmapsto \{x_1, \ldots, x_q\}$ $\longrightarrow$ so wedge product comes from symbol concatenation.

Moreover, the ramification filtration $(1+\pi^m i^* \mathbb{G}_{a,X}) \otimes (i^* j_* \mathbb{G}_m)^{\otimes(q-1)}$, $m \geq 1$ induces a decreasing filtration $U^m R^q\psi\Lambda(q)$, $m \geq 0$ (where $U^0 = R^q\psi\Lambda(q)$) of $R^q\psi\Lambda(q)$. Denote by $gr^m R^q\psi\Lambda(q) = U^m/U^{m+1}$.

Thm (Bloch–Kato)

(1) the symbol map is surjective.

(2) $gr^0 R^q\psi\Lambda(q) \xrightarrow{\sim} \nu(q) \oplus \nu(q-1)$

$\{x_1, \ldots, x_q\} \longmapsto \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}, 0\right)$, $x_i \in i^* \mathbb{G}_{m,X}$

$\{x_1, \ldots, x_{q-1}, \pi\} \longmapsto \left(0, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}\right)$

(3) $1 \leq m <$ ~~$\frac{ep}{p-1}$~~ , have well-defined

$$\Omega_Y^{q-1} \oplus \Omega_Y^{q-2} \longrightarrow U^m R^q\psi\Lambda(q) / U^{m+1} = gr^m$$

$$\left(x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0\right) \longmapsto \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-1}\}$$

$$\left(0, x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}}\right) \longmapsto \{1 + \tilde{x}\pi^m, \tilde{y}_1, \ldots, \tilde{y}_{q-2}, \pi\}$$

for $m < e'$, inducing $p \nmid m$, $\Omega_Y^{q-1} \xrightarrow{\sim} gr^m R^q\psi\Lambda(q)$

$p \mid m$, $d\Omega_Y^{q-1} \oplus d\Omega_Y^{q-2} \xrightarrow{\sim} gr^m R^q\psi\Lambda(q)$

(4) $U^{e'} R^q\psi\Lambda(q) = 0$

Hence, $R^q\psi\Lambda(q)$ admits a finite filtration s.t. $\left\{\begin{array}{l} gr^0 = \oplus \text{ of } \nu \\ U^1 = \text{successive } \mathbb{F}_p\text{-extensions loc. free. } \mathcal{O}_{Y_\bullet}\text{-mod of finite rk.}\end{array}\right.$

and $R^q\psi\Lambda(q) = 0$ if $q-1 > r$ ($\Leftrightarrow q > r+1$).

Hence $R\psi\Lambda(q) \in D^{[0, r+1]}$. (adjoining ζ_p, field extension - trace method).

Rmk 1) X can be only "pro-smooth".

2) If replacing the étale topology by the Zariski topology, $\varepsilon : Y_{et} \to Y_{Zar}$,

the results are the same for $R^q\psi_{Zar}\Lambda(q) := H^q(R\varepsilon_* R\psi\Lambda(q))$,

except that $U^{e'} = gr^{e'} \xleftarrow{\sim} \Omega_Y^{q-1} / (1+aC)Z_Y^{q-1} \oplus \Omega_Y^{q-2} / (1+aC) Z_Y^{q-2}$.

C the Cartier operator: $Z_Y^q \longrightarrow Z_Y^q / B_Y^q \xleftarrow{C^{-1}} \Omega_Y^{q-1}$

$$x^p \frac{dy_1}{y_1} \wedge \cdots \longleftarrow x \frac{dy_1}{y_1} \wedge \cdots$$

By adjoining the $(p-1)$th root of $a = \left(\frac{p}{\pi^e} \bmod \pi\right) \in k^\times$ with an unramified extension of K, one gets $(1+aC) Z_Y^{q-1} = (1-C) Z_Y^{q-1}$, since

$$\left(x^p + aCx^p = x^p + (a^{1/p-1})^{p-1}x\right.$$

$$= (-a^{1/p-1})^p\left[(-a^{-1/p-1}x)^p - (-a^{-1/p-1}x)\right] = (-a^{1/p-1})(1-C)(-a^{-1/p-1}x)^p\Big)$$

3) Also, there are natural pairings:

① étale: $(\nu(q) \oplus \nu(q-1)) \times (\nu(r+1-q) \oplus \nu(r-q)) \longrightarrow \nu(r)$, $\mathbb{F}_p$-perfect

$(\omega, \omega') \times (\tau, \tau') \longmapsto \pm\omega\wedge\tau' \pm \omega'\wedge\tau$.

② Zariski: (suppose $\zeta_p \in K \Rightarrow e' = \frac{ep}{p-1} \in p\mathbb{Z}$) for $s+t = r+2$

$$\Omega^{s-1} \times \Omega^{t-1} \xrightarrow{\text{wedge}} \Omega_Y^r, \text{ coherent duality.}$$

(x^p, z^p) x-linear?

$$(d\Omega_Y^{s-1} \oplus d\Omega_Y^{s-2}) \times (d\Omega_Y^{t-1} \oplus d\Omega_Y^{t-2}) \to \Omega_Y^r, \text{ coherent duality}$$

$(d\omega, d\omega') \times (d\tau, d\tau') \longrightarrow \pm\omega\wedge d\tau' \pm \omega'\wedge d\tau$.

4) the prove is local: they studied the stalks of $R^q\psi\Lambda(q)$

at the geometric point $\bar{y} \to Y \hookrightarrow X$ is $H^q_{ét}(\mathcal{O}^{sh}_{X,\bar{y}}, \Lambda(q))$.

Reason: any étale ring map lifts along surjection (or for schemes, closed imm.)

to an étale ring map.

As a consequence: using the canonical trace map $R\psi\Lambda(r+1) \to \mathrm{gr}^0 R^{r+1}\psi\Lambda(r+1)[-r-1]$

$$\wr$$

one construct a pairing $\nu(r)[-r-1]$.

$$R\psi\Lambda(s) \otimes^L R\psi\Lambda(t) \longrightarrow R\psi\Lambda(r+1) \xrightarrow{Tr} \nu(r)[-r-1] \text{ for } s+t=r+1.$$

Before stating our duality statement (which should be a modification the above pairing), we stress that the duality of [Kato, I] is need, so $Y_{ét}$ is not enough, have to consider Y_{RP}. Accordingly, whether a relatively perfect map lifts along surjection $(Y \hookrightarrow X)$ to a nice map to X should be a question.

2.2. The relatively perfectly smooth site Y_{RPS}.

we start with some results on Y_{RP}, then pass to Y_{RPS} since the ind-smoothness is in fact needed for applying Bloch-Kato's results.

Prop. 1) $Sch/_Y \to Sch/_Y$ has a right adjoint $T \longmapsto G(T)$.

$T \longmapsto T^{(p)}$ "Weil restriction along F_Y".

2) $Y_{RP} \hookrightarrow Sch/_Y$ admits right adjoint $(-)^{RP} = \varprojlim_n G^n(-)$

here the transition maps are given by $G(T) \xrightarrow[\text{rel. Frob}]{} G(T)^p \xrightarrow[\text{adjunction}]{} T$,

which are affine.

The smoothness on Y_{RP} should be the rel. perfection of smoothness:

Def. $\begin{matrix} Y' \\ \downarrow \\ Y \end{matrix}$ rel. perfectly smooth if Zar. locally $Y' = T^{RP}$ for some $\begin{matrix} T \\ \downarrow \\ Y \end{matrix}$ smooth.

$Y_{RPS} = \left(\text{perfectly smooth schemes}/Y\right)$ + ét. topology.

[Kato, I] applies to Y_{RPS} since it contains $\mathbb{G}_a = (\mathbb{A}^1_Y)^{RP}$.

__Example.__ $\mathbb{G}_a = (\mathbb{A}^1_Y)^{RP}$, explicit description. if $Y = \operatorname{Spec} R$ has a p-base.

$$(\mathbb{A}^1_Y)^{RP} = \varprojlim_n G^n(\mathbb{A}^1_Y).$$

suppose $\underset{=R}{\mathcal{O}(Y)}$ has a p-base $(b_1, \dots, b_r)$, $R \xrightarrow{(\cdot)^p} R$ with basis $\underline{b}^I$.

$$\operatorname{Hom}(R', G(\mathbb{A}^1_Y)) = \operatorname{Hom}(R^{(p)}, \mathbb{A}^1_Y) = R^{(p)} = R' \otimes_{R \nearrow F} R.$$

so. $G(\mathbb{A}^1_Y) \cong \mathbb{A}^{p^r}_Y$ $\qquad \cong \bigoplus_I R' \cdot \underline{b}^I \leftarrow \text{rank} = p^r.$

The transition maps under this choice $\qquad \uparrow$ non-canonical.

of p-base $\underline{b}$ is:

$$\begin{array}{ccc} \mathbb{A}^{p^r}_Y & \longrightarrow & \mathbb{A}^1_Y \\ \sum_I X_I^p \underline{b}^I & \longleftarrow\!\shortmid & X \end{array}$$

Similarly. $G^n(\mathbb{A}^1_Y) \longrightarrow G^{n-1}(\mathbb{A}^1_Y)$, so at the ~~sty~~ schematic level, $(\mathbb{A}^1_Y)^{RP}$ is a pro-smooth Y-scheme.

$\qquad \wr\!\shortparallel \qquad\qquad\qquad \wr\!\shortparallel$

$\qquad \mathbb{A}^{p^{nr}} \qquad\qquad \mathbb{A}^{p^{(n-1)r}}$

__Cor.__ May lift

$$\begin{array}{ccc} \mathbb{G}_a = (\mathbb{A}^1_Y)^{RP} & \hookrightarrow & \widetilde{\mathbb{G}}_a^{\tilde{\underline{b}}} \\ \downarrow & \square & \downarrow \\ Y & \hookrightarrow & X = \operatorname{Spec}(A) \end{array}$$

by choosing a lift ~~of~~ $\tilde{\underline{b}}$ $\underline{b}$ in A.

and $\widetilde{\mathbb{G}}_a^{\tilde{\underline{b}}} = \operatorname{Spec}(A')$ with $A' \leftarrow A$ ind-smooth!

2.3. Statement of the theorem of [Kato-Suzuki]

Thm. $\exists$ canonical trace map $R\Psi\Lambda(s) \otimes^L R\Psi\Lambda(t) \to \nu(r)[-r-1]$

for $s+t=r+1$, where $\Psi = i^* Rj_*$.

$$Y_{RPS} \overset{i}{\hookrightarrow} X_{RPS} \overset{j}{\hookleftarrow} U_{Et}, \quad X_{RPS} = \left(\begin{array}{ccc} T & \text{flat} & T\times_X Y \\ \downarrow & \text{st.} & \downarrow \text{ is RPS.} \\ X & & Y \end{array} \right)$$

$+$ étale topology

which is a perfect derived duality. $R\Psi\Lambda(s) \xrightarrow{\sim} R\mathcal{H}om_{Y_{RPS},\mathbb{F}_p}(\dots)$

2.4. Construction of the trace map.

The idea is to make Bloch-Kato applicable in our sites.

Lem 1 (canonical lifting)

setting: $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(R')$
$\downarrow$ rel. perfect
Y

Then $\exists!$ complete $\hat{A}$-alg. $R'_{\hat{A}}$ flat$/o_K$,

s.t. $R'_{\hat{A}}/\pi \cong R'$.

$$\begin{tikzcd} R'_{\hat{A}} \arrow[r] & R' \\ \hat{A} \arrow[u] & \\ A \arrow[u] \arrow[r] & R \arrow[uu, "\text{rel. perfect}"'] \\ o_K \arrow[u, "\text{flat}"] \arrow[r] \arrow[ur, phantom, "\square"] & k \arrow[u] \end{tikzcd}$$

In particular, any flat A-alg. whose mod π reduction is R' gives $R'_{\hat{A}}$ under π-adic completion.

Lem 2 (ind-smooth lifting)

$Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$

~~Same setting~~ $+$ R has a p-base, $y \to \mathrm{Spec}\, R$, $T \to Y$ smooth

$R' = \mathcal{O}_{T,?}$ or $\mathcal{O}^h_{T_{RP},y}$ or $\mathcal{O}^{sh}_{T_{RP},\bar{y}}$.

Then $\exists$ ind-smooth A-alg A' local or hens. local or sthens. local over $\mathcal{O}_{X,y}$ st. $A'/\pi \cong R'$. and A' is π-henselian.

In particular $\hat{A'} = R'_{\hat{A}}$.

pf. **Lem 1**.
$$\begin{tikzcd} W(R') \arrow[r] & ? \arrow[r] & R' \\ W(R) \arrow[u] \arrow[r] & \hat{A} \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$
take $? = (W(R') \otimes_{W(R)} \hat{A})^\wedge_\pi$.

Lem 2. Shrinking Spec A around y, may assume $R' = \mathcal{O}_{T^{RP}, y}$ (or $\cdots$) with

$T = \mathrm{Spec}(R_1)$, $R_1 \xleftarrow{\text{ét}} R[X_1, \ldots, X_m]$. Then reduce,

by compatibility of product, to $T = \mathrm{Spec}\, R[X] = \mathbb{A}^1_y$, so $T^{RP} = \mathbb{G}_m$,

in which case this is done by the above explicit example of $\mathbb{G}_m = (\mathbb{A}^1_y)^{RP}$.

Once this is done, for any torsion sheaf on U_{Et}, $y' \in Y_{RPS}$

$$(R^q\psi \mathcal{F})_{\bar{y} \to y'} \overset{\text{can}}{\cong} H^q_{\text{ét}}(R'_{\hat{A}}[\tfrac{1}{p}], \mathcal{F}) \underset{\substack{\uparrow \\ \text{Gabber-Fujiwara}}}{\cong} H^q_{\text{ét}}(A'[\tfrac{1}{p}], \mathcal{F})$$

where $R' = \mathcal{O}^{sh}_{Y', \bar{y}}$, $R'_{\hat{A}}$ the canonical lifting, A' the p-henselian

ind-smooth lifting ("decompletion of $R'_{\hat{A}}$"), respectly for

$R' = \mathcal{O}_{Y, y}$ $\cdots\cdots$ for $\mathcal{H}^q(R\varepsilon_* R\psi \mathcal{F})_y$ the Zariski version.

Hence Bloch-Kato's results on $gr^m R^q\psi\Lambda(q) <$ ét. topology / Zar. topology apply.

$\Rightarrow$ The pairing $R\psi\Lambda(s) \otimes^L R\psi\Lambda(t) \longrightarrow \nu(r)[-r-1]$ is constructed as usual.

Rmk. same statement for $\Lambda = \mathbb{Z}/p^n$, but need to replace $\nu(\cdot)$ by $\nu_n(\cdot) \subseteq W_n\Omega^{\bullet}_Y$.

3. The proof of [Kato-Suzuki] (may suppose $\zeta_p \in K$, $a^{1/p-1} \in k^\times$)

Rmk. Reduced to $n=1$ case i.e. $\Lambda = \mathbb{Z}/p$. since the extension deg is prime to p. since otherwise $U^{e'}$ is complicated $\Rightarrow$ trace method by some easy calculation + identification.

Consider the ~~complex~~ object $R\psi\Lambda \in D_{\psi}^{[0,r+1]}(Y_{RPS}, \mathbb{F}_p) \subseteq D(Y_{RPS}, \mathbb{F}_p)$.

Define the "intermediate" truncation

$$\to \tau^{\le t-1} \xrightarrow{U^1H^t[-t]} \tau_1^{\le t} \xrightarrow{gr^0H^t[-t]} \tau^{\le t} \xrightarrow{U^1H^{t+1}[-t]} \tau_1^{\le t+1} \xrightarrow{gr^0H^{t+1}[-t-1]} \tau_1^{\le t+1}$$

successive cofibers:

& $\to \tau^{\ge s} \xrightarrow{U^1H^s[-s]} \tau_1^{\ge s} \xrightarrow{gr^0H^s[-s]} \tau^{\ge s+1}$

successive fibers:

Prop. $s+t = r+1$. Then the pairing $R\psi\Lambda \otimes^L R\psi\Lambda \to \nu(r)[-r-1]$ induces uniquely

$$\begin{array}{ccccc}
\tau^{\ge s+1} & \otimes^L & \tau_1^{\le t} & \longrightarrow & \nu(r)[-r-1] \\
fib = gr^0H^s[-s] \uparrow & & \downarrow cofib = gr^0H^t[-t] & & \| \\
\tau_1^{\ge s} & \otimes^L & \tau^{\le t} & \longrightarrow & \nu(r)[-r-1] \\
fib = U^1H^s[-s] \uparrow & & \downarrow cofib = U^1H^{t+1}[-t-1] & & \| \\
\tau^{\ge s} & \otimes^L & \tau_1^{\le t+1} & \longrightarrow & \nu(r)[-r-1]
\end{array}$$

Compatible between them and as their fibers/cofibers:

$$\boxed{gr^0H^s \otimes^L gr^0H^t \longrightarrow \nu(r)}$$

$$\boxed{U^1H^s \otimes U^1H^{t+1}\Sigma \longrightarrow \nu(r)[1]}$$

Moreover, these fit into

$D(-) = R\mathcal{H}om_{Y_{RPS}}(-, \nu(r)[-r-1])$ on $D(Y_{RPS}, \mathbb{F}_p)$

$$\begin{array}{ccccc}
gr^0H^s[-s] & \to & \tau_1^{\ge s} & \to & \tau^{\ge s+1} \\
\downarrow & & \downarrow & & \downarrow \\
D(gr^0H^t[-t]) & \to & D(\tau^{\le t}) & \to & D(\tau_1^{\le t})
\end{array}
\quad \& \quad
\begin{array}{ccccc}
U^1H^s[-s] & \to & \tau^{\ge s} & \to & \tau_1^{\ge s} \\
\downarrow & & \downarrow & & \downarrow \\
D(U^1H^{t+1}[-t-1]) & \to & D(\tau_1^{\le t+1}) & \to & D(\tau^{\le t})
\end{array}$$

Pf Let's first show the first $\Uparrow$.

For degree reasons,

$$gr^0 H^s[-s+1] \longrightarrow R\mathcal{H}om(\tau_1^{\leq t}, \nu(r))[-r-1]$$

$\deg = s-1$ $\qquad$ $\deg \geq r+1-t = s$

is always zero,

$$\begin{tikzcd} gr^0 H^s[-s+1] \arrow[drr] & & \\ \tau^{\geq s+1} \arrow[u] \arrow[rr, "\exists!"] & & R\mathcal{H}om(\tau_1^{\leq t}, \nu(r))[-r-1] \\ \tau_1^{\geq s} \arrow[u] \arrow[urr] & & \\ gr^0 H^s[-s] \arrow[u] \arrow[uurr] & & \end{tikzcd}$$

$=0 \Rightarrow$ uniqueness

$=0 \Rightarrow$ existence

and $gr^0 H^s[-s] \longrightarrow R\mathcal{H}om(\tau_1^{\leq t}, \nu(r))[-r-1]$

factors through $gr^0 H^s[-s] \longrightarrow \mathcal{H}om(H^t(\tau_1^{\leq t}), \nu(r))[-r-1+t]$

$$\| $$

which vanishes by [Kato, I] $\qquad \mathcal{H}om(U^1 H^t, \nu(r))[-s]$

$\uparrow$ v. Bundle of type M^{RP} $\qquad$ $\uparrow$ $\mathbb{F}_p$

[Kato, I] $\Longrightarrow$ only $\mathcal{E}xt^1$ is non-zero

$R\mathcal{H}om(M^{RP}, \nu(r)) \simeq (M^\vee)^{RP}[-1]$

Now study each of the pairing $\square$, $\square$ above, we have

compatibilities:

$$\boxed{gr^0 H^s \otimes^L gr^0 H^t \longrightarrow \nu(r)}$$

which is perfect.

$$\begin{array}{ccccc} gr^0H^s & \otimes^L & gr^0H^t & \longrightarrow & \nu(r) \\ \wr\| & & \wr\| & & \| \\ (\nu(s)\oplus\nu(s-1)) & \otimes^L & (\nu(t)\oplus\nu(t-1)) & \longrightarrow & \nu(r) \end{array} \quad ①$$

and.

$$\cancel{R} \quad U^1/U^{e'} R^s\psi_{Zar}\Lambda \otimes^L U^1/U^{e'} R^{t+1}\psi_{Zar}\Lambda \otimes \longrightarrow U^{e'} R^{r+2}\psi_{Zar}\Lambda$$

$$\|$$

pieces $\rightsquigarrow$ $l+m = e'$, $l, m \geq 1$ $\qquad gr^l R^s\psi_{Zar}\Lambda \otimes^L gr^m R^{t+1}\psi_{Zar}\Lambda \longrightarrow \Omega^r_Y/(1-C)\Omega^r_Y$

$$\begin{array}{ccccc} \wr\| & & \wr\| & & \uparrow \text{quotient} \\ M & \otimes^L & M^\vee & \longrightarrow & \Omega^r_Y \end{array} \quad ②$$

It suffices to verify the second one is perfect.

(indeed, the compatibility of the second one is a symbol calculation that I'll fill out later on, and if it is perfect, then by adjunction $\varepsilon^* \vdash R\varepsilon_*$ and some easy identifications, and the following claim, we get a perfect pairing/duality:

$$\begin{tikzcd}
U^1/U^{e'} R^s\Psi\Lambda \otimes^L U^1/U' R^{t+1}\Psi\Lambda \arrow[r] & \nu(r)[1] \\
U^1 R^s\Psi\Lambda \otimes^L U^1 R^{t+1}\Psi\Lambda \arrow[u, "\cong"] \arrow[r] & \nu(r)[1] \arrow[u, equal]
\end{tikzcd}$$
)

<u>Claim</u>. $R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(M, \nu(r)) = 0$ $\forall M$. loc. free $\mathcal{O}_{Y_{Zar}}$-mod of fin. rk.

Admitting this, let's finish the proof.

By [Kato, I], $(M^\vee)^{RP} \xrightarrow{\sim} R\mathcal{H}om_{Y_{RPS}/Y'}(M^{RP}, \nu(r)[1])$.

By $R\varepsilon_*$ (using $R\varepsilon_*(M^\vee)^{RP} = R\varepsilon_*\varepsilon^*(M^\vee) \cong M^\vee$, $M^{RP} = \varepsilon^* M$), one gets

$$M^\vee \xrightarrow{\sim} R\varepsilon_* R\mathcal{H}om_{Y_{RPS}/Y'}(\varepsilon^* M, \nu(r)[1])$$

$$\cong R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(M, R\varepsilon_*\nu(r)[1])$$

$$\overset{\text{claim}}{\cong} R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(M, R^1\varepsilon_*\nu(r)\cancel{[1][-1]})$$

and it is easy to see it is of our form, because on the étale level, the mapping $(M^\vee)^{RP} \otimes^L M^{RP} \to \nu(r)[1]$ is given exactly by

$$\begin{tikzcd}
(M^\vee)^{RP} \arrow[r] \arrow[rr, bend right, "\cong"'] & \mathcal{H}om_{\mathbb{F}_p}(M^{RP}, \Omega^r_Y) \arrow[r, "\delta"] & \mathcal{E}xt^1_{\mathbb{F}_p}(M^{RP}, \nu(r)) \\
& & R\mathcal{H}om_{\mathbb{F}_p}(M^{RP}, \nu(r)[1]) \arrow[u, "\cong"]
\end{tikzcd}$$

pf. of claim.

~~① $\nu(r) = [\Omega^r \xrightarrow{C-1} \Omega^r]$ on étale site.~~

Recall [Eilenberg - Maclane . Breen - Deligne] resolution:

$\exists$ proj. functorial resolution $C_\bullet(M) \to M$

with each term is finite $\oplus$ of $\mathbb{Z}[M^m]$

$\Rightarrow$ $R\mathcal{H}om_{Y_{RPS,Zar}}(F, \nu(r))$ is totalization of $R\mathcal{H}om_{Y_{RPS,Zar}}(C_\bullet(F), \nu(r))$.

OK. It suffices to show then that

$$R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(\circ, \nu(r)) \xrightarrow{\sim} R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(\mathbb{G}_a, \nu(r))$$

and by BD resolution, it is totalization of something whose columns are finite direct $\oplus$ of $R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(\mathbb{F}_p[*], \nu(r)) \Rightarrow R\mathcal{H}om_{Y_{RPS,Zar}/Y'}(\mathbb{F}_p[\mathbb{G}_a^m], \nu(r))$

which is identified evidently with $R\Gamma_{Zar}(Y', \nu(r)) \longrightarrow R\Gamma_{Zar}(\mathbb{A}^m_{Y'}, \nu(r))$

which is in turn isomorphism by Zariski $\mathbb{A}^1$-invariance of $\nu(r)$.

Addendum:

Compatibility: $s+t = r+2$, Zariski topology

$l+m = e'$

$$\begin{tikzcd}
\mathrm{gr}^l H^s \arrow[r, phantom, "\times"] & \mathrm{gr}^m H^t \arrow[rr, "\text{cup-product}"] & & U^{e'} \\
d\Omega^{s-1}_Y \oplus \dots \arrow[u, "\cong"'] & \dots \oplus d\Omega^{t-2} \arrow[u, "\cong"'] \arrow[drr] & & \Omega^r_Y/(1-C)\Omega^r_Y \arrow[u, "\cong"'] \\
\Omega^{s-1}_Y \arrow[u, "d\omega"'] \arrow[r, phantom, "\times"] & \Omega^{t-2}_Y \arrow[u, "d\tau'"'] \arrow[rr, dashed] & & \Omega^r_Y \arrow[u]
\end{tikzcd}$$

$p \mid l$ $\qquad \omega \qquad\qquad \omega \wedge d\tau'$

$$\begin{array}{ccccc}
\{1+x\pi^l, y_1, \dots, y_{s-1}\} & \times & \{1+z_0\pi^m, z_1, \dots, z_{t-2}, \pi\} & \longmapsto & \{1+x\pi^l, \dots, 1+z_0\pi^m, \dots\}_\pi \\
 & & & & \| ? \\
 & & & & \pm\{1+xz_0\pi^{e'}, \dots, z_1, \dots\}_\pi \\
\updownarrow & & \updownarrow & & \updownarrow \\
d\left(x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{s-1}}{y_{s-1}}\right) & \times & d\left(z_0\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_{t-2}}{z_{t-2}}\right) & \longmapsto & \left(x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{s-1}}{y_{s-1}}\right) \\
 & & z_0 \in \mathcal{O}_Y^* & & \wedge\left(z_0\frac{dz_0}{z_0}\wedge\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_{t-2}}{z_{t-2}}\right)
\end{array}$$

Enough to show: for $x \in \mathcal{O}_Y^*$, $z \in \mathcal{O}_Y^*$, that

$$\{1+x\pi^l, 1+z\pi^m, \pi\} = \pm\{1\pm xz\pi^{l+m}, z, \pi\} \pmod{U^{l+m+1}}$$

(in $K_2^M(i^*j_*\mathbb{G}_m)$)

$p \nmid l$, similarly, enough to show, for $x, z \in \mathcal{O}_Y^*$, that

$$\widehat{\{1+x\pi^l, 1+z\pi^m\}} = \pm\widehat{\{1\pm xz\pi^{l+m}, \pi\}} \pmod{U^{l+m+1}}$$

(in $\widehat{K^M_{r+2}(i^*j_*\mathbb{G}_m)}$)

Pf. By symbol calculation, for $x, z \in \mathcal{O}_Y^*$

$$\{1+x\pi^l, 1+z\pi^m\} \equiv \{1+x\pi^l, 1+z\pi^m(1+x\pi^l)\} \pmod{U^{(l+m)+l}}$$

$$= -\{-z\pi^m, 1+z\pi^m(1+x\pi^l)\}$$

$$= -\{-z\pi^m, 1+\frac{xz\pi^{l+m}}{1+z\pi^m}\}$$

$$= -\{-z\pi^m, 1+xz\pi^{l+m}\} \pmod{U^{(l+m)+m}}$$

since

$$1+\frac{xz\pi^{l+m}}{1+z\pi^m} = 1+xz\pi^{l+m}\left(1-\frac{z\pi^m}{1+z\pi^m}\right)$$

$$\{1+xz\pi^{l+m}, -z\pi^m\}$$

Hence if $p \mid l$.

$$\{1+x\pi^l, 1+z\pi^m, \pi\} \equiv \{1+xz\pi^{l+m}, -z\pi^m, \pi\} \pmod{U^{l+m+1}}$$

$$= \{1+xz, +z, \pi\} \quad \text{since } \{\pi,\pi\}=0. \quad \{-\pi,\pi\}=0$$

If $p \nmid l$, since $\zeta_p \in K \Rightarrow e' = \frac{e}{p-1}\cdot p \in p\mathbb{Z}$. so $p \nmid m$, $m \in \mathbb{F}_p^\times$

$$\{1+x\pi^l, \underbrace{\_\_}_{s-1}, 1+z\pi^m, \underbrace{\_\_}_{t-1}\} \equiv \{1+xz\pi^{l+m}, \_\_, -z\pi^m, \_\_\} \pmod{U^{l+m+1}}$$

$$= \{1+xz\pi^{l+m}, \underbrace{\_\_}_{s-1}, \pi^m, \underbrace{\_\_}_{t-1}\}$$

$$= (-1)^{t-1} m \{1+xz\pi^{l+m}, \underbrace{\_\_}_{s-1}, \underbrace{\_\_}_{t-1}, \pi\}, \quad m \in \mathbb{F}_p^\times.$$

since $\{\underbrace{\Theta_{m,Y}, \_\_, \Theta_{m,Y}}_{s+t=r+2 \text{ times}}\}$

lies in the image of $\Omega_Y^{(r+2)-1}$, which vanishes due to degree reasons.